

Weights and conservativity

by

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Abstract

The purpose of this article is to study conservativity in the context of triangulated categories equipped with a weight structure. As application, we establish (weight) conservativity for the restriction of the (generic) ℓ -adic realization to the category of motives of Abelian type of characteristic zero.

Keywords: weight structures, conservativity, weight conservativity, realizations, relative motives of Abelian type.

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0 Introduction

The aim of this article is to provide a proof of *weight conservativity* of the restriction of the (generic) ℓ -adic realization R_ℓ to the category of *motives of Abelian type* of characteristic zero (Theorem 3.4).

Here, weight conservativity refers to the following refinement of the usual categorical notion of conservativity: the functor R_ℓ detects not only isomorphisms, but also weights (and hence, their absence). In other words, a motive M of Abelian type *is without weights* $\alpha, \alpha + 1, \dots, \beta$, for integers $\alpha \leq \beta$, in the sense of [W1, Def. 1.10] if and only if the same is true for $R_\ell(M)$. This provides the main motivation for our study: as shown in [loc. cit.], absence of certain weights in the *boundary motive* of a scheme X allows for the construction of its *interior motive*.

Weight conservativity for motives of Abelian type over a point was previously established in [W3, Sect. 1]. This special case suffices *e.g.* for the analysis of weights in the boundary motive of *Picard surfaces* [W3, Sect. 3], essentially because the complement of the latter in their *Baily–Borel compactifications* is of dimension zero. The study of the boundary motive of *Shimura varieties*, whose boundary is higher dimensional, requires a version of weight conservativity for motives over higher dimensional bases, hence the need for Theorem 3.4. For an application to the case of *Siegel threefolds*, we refer to [W5].

In order to prove Theorem 3.4 and the intermediate conservativity results leading up to it (Theorems 2.11 and 3.3), it turns out that the notion of *weight structure* [B1] is central. In fact, the formal structure of the proofs is best understood in that abstract setting. This explains the title of the present work, and also, its organization. Section 1 is entirely situated in the context of functors r , whose source is a triangulated category \mathcal{C} equipped with a weight structure w . The following question appears natural: assuming that the weight structure is *bounded*, and r is *weight exact*, does conservativity of the restriction of r to the *heart* $\mathcal{C}_{w=0}$ imply (weight) conservativity of r ?

The analogous question for t -structures can be asked; there, the answer is obviously positive. Weight structures being much less rigid than t -structures, it is not surprising to see that additional hypotheses are needed. The main such condition concerns the heart: all our abstract results (Theorems 1.5, 1.8, 1.10 and 1.11) require $\mathcal{C}_{w=0}$ to be *semi-primary* [AK, Déf. 2.3.1]. The reason is that all proofs make a systematic use of the *minimal weight filtration* [W4, Sect. 2], whose existence is guaranteed only if $\mathcal{C}_{w=0}$ is semi-primary.

This explains why we restrict our attention to motives of Abelian type: while relative Chow motives may always be expected to form a semi-primary category [W4, Conj. 3.4], we are at present far from having a general proof at our disposal. Section 2 therefore sets up a precise motivic setting in which semi-primality is guaranteed (Theorem 2.10).

Section 3 then leads up to Theorem 3.4.

One word about the Hodge theoretical picture: an analogue of our main result Theorem 3.4 should certainly hold for the (generic) *Hodge theoretical realization* $R_{\mathbf{H}}$. Unfortunately, this realization is at present not fully available. Note however that [I] provides a Hodge theoretical realization for schemes which are smooth over \mathbb{C} . For the applications we have in mind [W5], it will however be necessary to consider singular schemes. Similarly, we need the compatibility of $R_{\mathbf{H}}$ with the functors f^* , f_* , $f_!$, $f^!$. In the present work, we thus replace $R_{\mathbf{H}}$ by the (generic) *Betti realization*, the price to be paid being that we *a priori* lose the intrinsic notion of weights on the target of the realization. Nonetheless, we have sufficient control on the situation to prove the analogue of Theorem 3.3: as R_{ℓ} , the restriction of the (generic) Betti realization to the category of motives of Abelian type is conservative.

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Conventions: Throughout the article, F denotes a finite direct product of fields of characteristic zero. We fix a base scheme \mathbb{B} , which is of finite type over some excellent scheme of dimension at most two. By definition, *schemes* are \mathbb{B} -schemes which are separated and of finite type (in particular, they are excellent, and Noetherian of finite dimension), *morphisms* between schemes are separated morphisms of \mathbb{B} -schemes, and a scheme is *nilregular* if the underlying reduced scheme is regular in the usual sense.

We use the triangulated, \mathbb{Q} -linear categories $DM_{\mathbb{B},c}(X)$ of *constructible Beilinson motives* over X [CD1, Def. 15.1.1], indexed by schemes X (always

in the sense of the above conventions). In order to have an F -linear theory at one's disposal, one re-does the construction, but using F instead of \mathbb{Q} as coefficients [CD1, Sect. 15.2.5]. This yields triangulated, F -linear categories $DM_{B,c}(X)_F$ satisfying the F -linear analogues of the properties of $DM_{B,c}(X)$. In particular, these categories are pseudo-Abelian (see [H, Sect. 2.10]). Furthermore, the canonical functor $DM_{B,c}(X) \otimes_{\mathbb{Q}} F \rightarrow DM_{B,c}(X)_F$ is fully faithful [CD1, Sect. 14.2.20]. As in [CD1], the symbol $\mathbf{1}_X$ is used to denote the unit for the tensor product in $DM_{B,c}(X)_F$. We shall employ the full formalism of six operations developed in [loc. cit.]. The reader may choose to consult [H, Sect. 2] or [W2, Sect. 1] for concise presentations of this formalism.

1 Conservativity and weight conservativity

We make free use of the terminology of and basic results on weight structures [B2, Sect. 1.3].

Definition 1.1. Let $r : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ be an F -linear exact functor between F -linear triangulated categories equipped with weight structures $(\mathcal{C}_{1,w \leq 0}, \mathcal{C}_{1,w \geq 0})$ and $(\mathcal{C}_{2,w \leq 0}, \mathcal{C}_{2,w \geq 0})$.

(a) The functor r is said to be *weight exact* if

$$r(\mathcal{C}_{1,w \leq 0}) \subset \mathcal{C}_{2,w \leq 0} \quad \text{and} \quad r(\mathcal{C}_{1,w \geq 0}) \subset \mathcal{C}_{2,w \geq 0}.$$

(b) If r is weight exact, then we denote by $r_{w=0}$ the functor

$$\mathcal{C}_{1,w=0} \longrightarrow \mathcal{C}_{2,w=0}$$

induced by the restriction of r to the heart $\mathcal{C}_{1,w=0}$.

Lemma 1.2. *Let $r : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ be an F -linear exact functor between F -linear triangulated categories equipped with weight structures. We assume the following.*

(1) *The functor r is weight exact.*

(2) *The functor $r_{w=0} : \mathcal{C}_{1,w=0} \rightarrow \mathcal{C}_{2,w=0}$ is full.*

Then for any integer n , and any two objects $X \in \mathcal{C}_{1,w \leq n}$ and $Z \in \mathcal{C}_{1,w \geq n}$, the map

$$r : \mathrm{Hom}_{\mathcal{C}_1}(X, Z) \longrightarrow \mathrm{Hom}_{\mathcal{C}_2}(r(X), r(Z))$$

is surjective.

Proof. Fix weight filtrations

$$X_{\leq n-1} \longrightarrow X \longrightarrow X_n \longrightarrow X_{\leq n-1}[1]$$

and

$$Z_n \longrightarrow Z \longrightarrow Z_{\geq n+1} \longrightarrow Z_n[1]$$

of X and Z , respectively, with $X_n, Z_n \in \mathcal{C}_{1,w=n}$, $X_{\leq n-1} \in \mathcal{C}_{1,w \leq n-1}$, and $Z_{\geq n+1} \in \mathcal{C}_{1,w \geq n+1}$. Given assumption (1), their images under r are weight filtrations of $r(X)$ and $r(Z)$ of the same type.

Let $\beta : r(X) \rightarrow r(Z)$. By orthogonality for the weight structure on \mathcal{C}_2 , the morphism β factors through a morphism $\beta' : r(X_n) \rightarrow r(Z_n)$. The shift by $[-n]$ of the latter gives $\beta'[-n] : r(X_n)[-n] \rightarrow r(Z_n)[-n]$, a morphism in $\mathcal{C}_{2,w=0}$. Given assumption (2), the morphism $\beta'[-n]$ lies in the image of r . But then so do β' and β . **q.e.d.**

Before stating our first abstract result on conservativity, let us recall an important notion.

Definition 1.3. Let \mathcal{C} be an F -linear triangulated category equipped with a weight structure w . Let $M \in \mathcal{C}$, and $n \in \mathbb{Z}$. A *minimal weight filtration concentrated at n* of M is a weight filtration

$$M_{\leq n-1} \longrightarrow M \longrightarrow M_{\geq n} \xrightarrow{\delta} M_{\leq n-1}[1]$$

($M_{\leq n-1} \in \mathcal{C}_{w \leq n-1}$, $M_{\geq n} \in \mathcal{C}_{w \geq n}$) such that the morphism δ belongs to the *radical* [AK, Déf. 1.4.1] of \mathcal{C} :

$$\delta \in \text{rad}_{\mathcal{C}}(M_{\geq n}, M_{\leq n-1}[1]) .$$

Any two minimal weight filtrations of the same object M are related by an isomorphism (which in general is *not* unique) [W4, proof of Thm. 2.2 (b)].

Proposition 1.4. *Let $r : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ be an F -linear exact functor between F -linear triangulated categories equipped with weight structures. We assume the following.*

- (1) *The functor r is weight exact.*
- (2) *The functor $r_{w=0} : \mathcal{C}_{1,w=0} \rightarrow \mathcal{C}_{2,w=0}$ is full.*

Then r maps minimal weight filtrations to minimal weight filtrations.

Proof. Let

$$(*) \quad M_{\leq n-1} \longrightarrow M \longrightarrow M_{\geq n} \xrightarrow{\delta} M_{\leq n-1}[1]$$

be a minimal weight filtration concentrated at n of $M \in \mathcal{C}_1$. Thanks to assumption (1), the exact triangle $r(*)$ is a weight filtration of $r(M)$.

According to Lemma 1.2, any morphism $r(M_{\leq n-1}[1]) \rightarrow r(M_{\geq n})$ lies in the image of r . Using this information, and applying the definition, one sees that $r(\delta)$ belongs to the radical of \mathcal{C}_2 since δ belongs to the radical of \mathcal{C}_1 . Therefore, $r(*)$ is a minimal weight filtration concentrated at n of $r(M)$. **q.e.d.**

Theorem 1.5. *Let $r : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ be an F -linear exact functor between F -linear triangulated categories equipped with weight structures. We assume the following.*

- (1) *The weight structure on \mathcal{C}_1 is bounded, i.e., its heart $\mathcal{C}_{1,w=0}$ generates \mathcal{C}_1 as a triangulated category.*
- (2) *The heart $\mathcal{C}_{1,w=0}$ is semi-primary [AK, Déf. 2.3.1] and pseudo-Abelian.*
- (3) *The functor r is weight exact.*
- (4) *The functor $r_{w=0}$ is full.*
- (5) *The functor $r_{w=0}$ is conservative.*

Then r is conservative.

Proof. r being an exact functor between triangulated categories, it suffices to show that only the zero object is mapped to zero under r . Thus, let $M \in \mathcal{C}_1$, and assume that $r(M) = 0$.

Given assumption (1), the category \mathcal{C}_1 is pseudo-Abelian if and only if its heart $\mathcal{C}_{1,w=0}$ is [B1, Lemma 5.2.1]. Given assumption (2), we are thus in the abstract situation studied in [W4, Sect. 2], meaning in particular that minimal weight filtrations *do* exist for any object of \mathcal{C}_1 [W4, Thm. 2.2 (a)]. Let

$$(*) \quad M_{\leq -1} \longrightarrow M \longrightarrow M_{\geq 0} \xrightarrow{\delta} M_{\leq -1}[1]$$

be a minimal weight filtration of M , i.e., $M_{\leq -1} \in \mathcal{C}_{1,w \leq -1}$, $M_{\geq 0} \in \mathcal{C}_{1,w \geq 0}$, and the morphism δ belongs to the radical of \mathcal{C}_1 .

On the one hand, the triangle $r(*)$ is exact, and $r(M) = 0$. Therefore, the morphism $r(\delta)$ is an isomorphism.

On the other hand, according to Proposition 1.4 (applicable thanks to hypotheses (3) and (4)), $r(\delta)$ belongs to the radical of \mathcal{C}_2 .

But then $r(\delta)$ must be the zero morphism $0 \rightarrow 0$, meaning that both $M_{\leq -1}$ and $M_{\geq 0}$ are mapped to zero under r .

According to hypothesis (1), the weight structure on \mathcal{C}_1 is bounded. Therefore, the above procedure, successively applied to minimal weight filtrations concentrated at integers different from zero, allows to reduce to the case where M is pure of some weight, say n . But then $M[-n]$ belongs to $\mathcal{C}_{1,w=0}$. Now apply assumption (5). **q.e.d.**

Remark 1.6. The analogue of Theorem 1.5 for triangulated categories equipped with t -structures holds, and requires only the analogues of assumptions (1), (3), and (5).

Recall the following.

Definition 1.7 ([W1, Def. 1.6, Def. 1.10]). Let \mathcal{C} be an F -linear triangulated category equipped with a weight structure w . Let $M \in \mathcal{C}$, and $\alpha \leq \beta$ two integers (which may be identical).

(a) A *weight filtration of M avoiding weights $\alpha, \alpha + 1, \dots, \beta$* is an exact triangle

$$M_{\leq \alpha-1} \longrightarrow M \longrightarrow M_{\geq \beta+1} \longrightarrow M_{\leq \alpha-1}[1]$$

in \mathcal{C} , with $M_{\leq \alpha-1} \in \mathcal{C}_{w \leq \alpha-1}$ and $M_{\geq \beta+1} \in \mathcal{C}_{w \geq \beta+1}$.

(b) We say that $M \in \mathcal{C}$ *does not have weights $\alpha, \alpha + 1, \dots, \beta$* , or that M *is without weights $\alpha, \alpha + 1, \dots, \beta$* , if it admits a weight filtration avoiding weights $\alpha, \alpha + 1, \dots, \beta$.

We leave it to the reader to verify that any weight filtration avoiding weights $\alpha, \alpha + 1, \dots, \beta$ is a minimal weight filtration concentrated at n , for any integer between α and $\beta + 1$.

The following result sharpens Theorem 1.5; it will however not be needed in the sequel.

Theorem 1.8. *Let $r : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ be an F -linear exact functor between F -linear triangulated categories equipped with weight structures. The assumptions (1)–(5) are the same as in Theorem 1.5. Then r is weight conservative, i.e., it detects weights. More precisely, let $M \in \mathcal{C}_1$, and $\alpha \leq \beta$ two integers.*

- (a) *M lies in the heart $\mathcal{C}_{1,w=0}$ if and only if $r(M)$ lies in the heart $\mathcal{C}_{2,w=0}$.*
- (b) *M lies in $\mathcal{C}_{1,w \leq \alpha}$ if and only if $r(M)$ lies in $\mathcal{C}_{2,w \leq \alpha}$.*
- (c) *M lies in $\mathcal{C}_{1,w \geq \beta}$ if and only if $r(M)$ lies in $\mathcal{C}_{2,w \geq \beta}$.*
- (d) *M is without weights $\alpha, \alpha + 1, \dots, \beta$ if and only if $r(M)$ is without weights $\alpha, \alpha + 1, \dots, \beta$.*

Proof. According to assumption (1), the weight structure on \mathcal{C}_1 is bounded, and the “only if” parts of statements (a)–(d) are true. It therefore suffices to prove the “if” part of statement (d).

Consider minimal weight filtrations concentrated at weight α and at weight $\beta + 1$, respectively:

$$M_{\leq \alpha-1} \longrightarrow M \longrightarrow M_{\geq \alpha} \xrightarrow{\delta_\alpha} M_{\leq \alpha-1}[1],$$

$$M_{\leq \beta} \longrightarrow M \longrightarrow M_{\geq \beta+1} \xrightarrow{\delta_{\beta+1}} M_{\leq \beta}[1].$$

By orthogonality, the identity on M extends to a morphism of exact triangles

$$\begin{array}{ccccccc} M_{\leq \alpha-1} & \longrightarrow & M & \longrightarrow & M_{\geq \alpha} & \xrightarrow{\delta_\alpha} & M_{\leq \alpha-1}[1] \\ m \downarrow & & \parallel & & \downarrow & & m[1] \downarrow \\ M_{\leq \beta} & \longrightarrow & M & \longrightarrow & M_{\geq \beta+1} & \xrightarrow{\delta_{\beta+1}} & M_{\leq \beta}[1] \end{array}$$

By Proposition 1.4, the image under r of the above is a morphism relating minimal weight filtrations concentrated at weight α and at weight $\beta + 1$, respectively:

$$\begin{array}{ccccccc} r(M_{\leq \alpha-1}) & \longrightarrow & r(M) & \longrightarrow & r(M_{\geq \alpha}) & \xrightarrow{r(\delta_\alpha)} & r(M_{\leq \alpha-1}[1]) \\ r(m) \downarrow & & \parallel & & \downarrow & & r(m)[1] \downarrow \\ r(M_{\leq \beta}) & \longrightarrow & r(M) & \longrightarrow & r(M_{\geq \beta+1}) & \xrightarrow{r(\delta_{\beta+1})} & r(M_{\leq \beta}[1]) \end{array}$$

Thus, both $r(\delta_\alpha)$ and $r(\delta_{\beta+1})$ lie in the radical of \mathcal{C}_2 .

But $r(M)$ is supposed to be without weights $\alpha, \alpha + 1, \dots, \beta$, meaning that there is a minimal weight filtration of $r(M)$, which is concentrated at any integer between α and $\beta + 1$. Unicity of minimal weight filtrations in \mathcal{C}_2 [W4, proof of Thm. 2.2 (b)] shows that this latter weight filtration is isomorphic to both the top and the bottom of the above diagram, meaning in particular that they are (abstractly) isomorphic to each other. In particular, the object $r(M_{\leq \beta})$ belongs to $\mathcal{C}_{2, w \leq \alpha-1}$. Orthogonality then allows to extend the identity on $r(M)$ to a morphism of exact triangles

$$\begin{array}{ccccccc} r(M_{\leq \beta}) & \longrightarrow & r(M) & \longrightarrow & r(M_{\geq \beta+1}) & \xrightarrow{r(\delta_{\beta+1})} & r(M_{\leq \beta}[1]) \\ n \downarrow & & \parallel & & \downarrow & & n[1] \downarrow \\ r(M_{\leq \alpha-1}) & \longrightarrow & r(M) & \longrightarrow & r(M_{\geq \alpha}) & \xrightarrow{r(\delta_\alpha)} & r(M_{\leq \alpha-1}[1]) \end{array}$$

Using the fact that both $r(\delta_\alpha)$ and $r(\delta_{\beta+1})$ lie in the radical, one shows that both compositions $n \circ r(m)$ and $r(m) \circ n$ are automorphisms; in particular, $r(m)$ is an isomorphism.

But then (Theorem 1.5), so is m itself. This yields a weight filtration

$$M_{\leq \alpha-1} \longrightarrow M \longrightarrow M_{\geq \beta+1} \longrightarrow M_{\leq \alpha-1}[1]$$

of M avoiding weights $\alpha, \alpha + 1, \dots, \beta$.

q.e.d.

The second half of the present section treats conservativity in a slightly different context. The source of the functors in question remains equipped with a weight structure. But their target is only supposed to be Abelian. Recall [B1, Prop. 2.1.2 1] that any (covariant) additive functor \mathcal{H} from a triangulated category \mathcal{C} carrying a weight structure w , to an Abelian category \mathfrak{A} admits a canonical *weight filtration* by sub-functors

$$\dots \subset W_n \mathcal{H} \subset W_{n+1} \mathcal{H} \subset \dots \subset \mathcal{H}.$$

According to [B1, Def. 2.1.1] (use the normalization of [B2, Def. 1.3.1] for the signs of the weights), for an object M of \mathcal{C} , and $n \in \mathbb{Z}$, the sub-object $W_n \mathcal{H}(M) \subset \mathcal{H}(M)$ is defined as the image of the morphism $\mathcal{H}(\iota_{w \leq n})$, for *any* weight filtration

$$M_{w \leq n} \xrightarrow{\iota_{w \leq n}} M \longrightarrow M_{w \geq n+1} \longrightarrow M_{w \leq n}[1]$$

(with $M_{w \leq n} \in \mathcal{C}_{w \leq n}$ and $M_{w \geq n+1} \in \mathcal{C}_{w \geq n+1}$). For any $m \in \mathbb{Z}$, one defines

$$\mathcal{H}^m : \mathcal{C} \longrightarrow \mathfrak{A}, \quad M \longrightarrow \mathcal{H}(X[m]);$$

according to the usual convention, the weight filtration of $\mathcal{H}^m(M)$ equals the weight filtration of $\mathcal{H}(X[m])$, i.e., it differs by *décalage* from the intrinsic weight filtration of the covariant additive functor \mathcal{H}^m .

The proof of the following is formal, and therefore left to the reader (cmp. [W3, Lemma 1.11]).

Lemma 1.9. *Let $\mathcal{H} : \mathcal{C}_1 \rightarrow \mathcal{C}_3$ be an F -linear functor between F -linear categories, \mathcal{C}_1 being equipped with a weight structure. We assume the following.*

- (1) *Any morphism in the image of \mathcal{H} is strict with respect to the weight filtration of \mathcal{H} .*
- (2) *The restriction of \mathcal{H} to the heart $\mathcal{C}_{1,w=0}$ maps the radical to zero.*

Then for any integer n , and any two objects $X \in \mathcal{C}_{1,w \leq n}$ and $Z \in \mathcal{C}_{1,w \geq n}$, the map

$$\mathcal{H} : \mathrm{Hom}_{\mathcal{C}_1}(Z, X) \longrightarrow \mathrm{Hom}_{\mathcal{C}_3}(\mathcal{H}(Z), \mathcal{H}(X))$$

maps the radical to zero.

Theorem 1.10. *Let $\mathcal{H} : \mathcal{C}_1 \rightarrow \mathcal{C}_3$ be an F -linear homological functor between F -linear categories, \mathcal{C}_1 being equipped with a weight structure. We assume the following.*

- (1) *The weight structure on \mathcal{C}_1 is bounded.*
- (2) *The heart $\mathcal{C}_{1,w=0}$ is semi-primary and pseudo-Abelian.*
- (3) *Any morphism in the image of \mathcal{H} is strict with respect to the weight filtration of \mathcal{H} .*
- (4) *The restriction of \mathcal{H} to the heart $\mathcal{C}_{1,w=0}$ maps the radical to zero.*
- (5) *Zero is the only object of the heart $\mathcal{C}_{1,w=0}$ mapped to zero under (the restriction to $\mathcal{C}_{1,w=0}$ of) the functor $(\mathcal{H}^m)_{m \in \mathbb{Z}}$.*

Then $(\mathcal{H}^m)_{m \in \mathbb{Z}}$ is conservative.

Proof. Let $M \in \mathcal{C}_1$. Let

$$(*) \quad M_{\leq -1} \longrightarrow M \longrightarrow M_{\geq 0} \xrightarrow{\delta} M_{\leq -1}[1]$$

be a minimal weight filtration of M (here, we use assumptions (1) and (2)). According to Lemma 1.9 (applicable thanks to hypotheses (3) and (4)), $\mathcal{H}(\delta) = 0$. The functor \mathcal{H} is supposed to be homological, hence if $\mathcal{H}(M)$

is zero, then both $\mathcal{H}(M_{\leq -1})$ and $\mathcal{H}(M_{\geq 0})$ are zero. Similarly, if $\mathcal{H}^m(M)$ is zero for some $m \in \mathbb{Z}$, then so are $\mathcal{H}^m(M_{\leq -1})$ and $\mathcal{H}^m(M_{\geq 0})$.

Now let $M \in \mathcal{C}_1$, and assume that $\mathcal{H}^m(M) = 0$ for all $m \in \mathbb{Z}$. We need to show that $M = 0$. According to hypothesis (1), the weight structure on \mathcal{C}_1 is bounded. Therefore, the above procedure, successively applied to minimal weight filtrations concentrated at integers different from zero, allows to reduce to the case where M is pure of some weight, say n . But then $M[-n]$ belongs to $\mathcal{C}_{1,w=0}$. Now apply assumption (5). **q.e.d.**

As before, there is a version “with weights” of the above.

Theorem 1.11. *Let $\mathcal{H} : \mathcal{C}_1 \rightarrow \mathcal{C}_3$ be an F -linear homological functor between F -linear categories, \mathcal{C}_1 being equipped with a weight structure. The assumptions (1)–(5) are the same as in Theorem 1.10. Then $(\mathcal{H}^m)_{m \in \mathbb{Z}}$ is weight conservative, i.e., it detects weights. More precisely, let $M \in \mathcal{C}_1$, and $\alpha \leq \beta$ two integers.*

- (a) *M lies in the heart $\mathcal{C}_{1,w=0}$ if and only if $\mathcal{H}^n(M)$ is pure of weight n , for all $n \in \mathbb{Z}$.*
- (b) *M lies in $\mathcal{C}_{1,w \leq \alpha}$ if and only if $\mathcal{H}^n(M)$ is of weights $\leq n + \alpha$, for all $n \in \mathbb{Z}$.*
- (c) *M lies in $\mathcal{C}_{1,w \geq \beta}$ if and only if $\mathcal{H}^n(M)$ is of weights $\geq n + \beta$, for all $n \in \mathbb{Z}$.*
- (d) *M is without weights $\alpha, \alpha + 1, \dots, \beta$ if and only if $\mathcal{H}^n(M)$ is without weights $n + \alpha, n + \alpha + 1, \dots, n + \beta$, for all $n \in \mathbb{Z}$.*

Proof. According to assumption (1), the weight structure on \mathcal{C}_1 is bounded, and the “only if” parts of statements (a)–(d) are true. It therefore suffices to prove the “if” part of statement (d).

Consider minimal weight filtrations concentrated at weight α and at weight $\beta + 1$, respectively:

$$M_{\leq \alpha-1} \longrightarrow M \longrightarrow M_{\geq \alpha} \xrightarrow{\delta_\alpha} M_{\leq \alpha-1}[1],$$

$$M_{\leq \beta} \longrightarrow M \longrightarrow M_{\geq \beta+1} \xrightarrow{\delta_{\beta+1}} M_{\leq \beta}[1].$$

By orthogonality, the identity on M extends to a morphism of exact triangles

$$\begin{array}{ccccccc} M_{\leq \alpha-1} & \longrightarrow & M & \longrightarrow & M_{\geq \alpha} & \xrightarrow{\delta_\alpha} & M_{\leq \alpha-1}[1] \\ m \downarrow & & \parallel & & \downarrow & & m[1] \downarrow \\ M_{\leq \beta} & \longrightarrow & M & \longrightarrow & M_{\geq \beta+1} & \xrightarrow{\delta_{\beta+1}} & M_{\leq \beta}[1] \end{array}$$

By Lemma 1.9, both $\mathcal{H}^*(\delta_\alpha)$ and $\mathcal{H}^*(\delta_{\beta+1})$ are zero. Thus, the above morphism of exact triangles induces a morphism of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{H}^*(M_{\leq \alpha-1}) & \longrightarrow & \mathcal{H}^*(M) & \longrightarrow & \mathcal{H}^*(M_{\geq \alpha}) \longrightarrow 0 \\ & & \mathcal{H}^*(m) \downarrow & & \parallel & & \downarrow \\ 0 & \longrightarrow & \mathcal{H}^*(M_{\leq \beta}) & \longrightarrow & \mathcal{H}^*(M) & \longrightarrow & \mathcal{H}^*(M_{\geq \beta+1}) \longrightarrow 0 \end{array}$$

Our hypothesis on weights avoided in $\mathcal{H}^*(M)$ implies that the monomorphism $\mathcal{H}^*(m)$ is in fact an isomorphism.

But then (Theorem 1.10), so is m itself. This yields a weight filtration

$$M_{\leq \alpha-1} \longrightarrow M \longrightarrow M_{\geq \beta+1} \longrightarrow M_{\leq \alpha-1}[1]$$

of M avoiding weights $\alpha, \alpha+1, \dots, \beta$.

q.e.d.

2 Relative motives of Abelian type

Let S be a scheme (in the sense of the conventions fixed in our Introduction).

Definition 2.1. A *good stratification* of S indexed by a finite set \mathfrak{S} is a collection of locally closed sub-schemes S_σ indexed by $\sigma \in \mathfrak{S}$, such that

$$S = \coprod_{\sigma \in \mathfrak{S}} S_\sigma$$

on the set-theoretic level, and such that the closure $\overline{S_\sigma}$ of any stratum S_σ is a union of strata S_τ .

In the setting of Definition 2.1, we shall often write $S(\mathfrak{S})$ instead of S . Recall the following result.

Theorem 2.2 ([W4, Thm. 4.5 (a)]). *Let $S(\mathfrak{S}) = \coprod_{\sigma \in \mathfrak{S}} S_\sigma$ be a good stratification of a scheme $S(\mathfrak{S})$. Assume the following for all $\sigma \in \mathfrak{S}$: (α) the stratum S_σ is nilregular, (β) for any stratum $i_\tau : S_\tau \hookrightarrow \overline{S_\sigma}$ contained in the closure $\overline{S_\sigma}$ of S_σ , the functor*

$$i_\tau^! : DM_{\mathbb{B},c}(\overline{S_\sigma})_F \longrightarrow DM_{\mathbb{B},c}(S_\tau)_F$$

maps $\mathbf{1}_{\overline{S_\sigma}}$ to a Tate motive over S_τ . Then the categories $DMT(S_\sigma)_F$ of Tate motives over S_σ [L2, Sect. 3.3], $\sigma \in \mathfrak{S}$, can be glued to give a full, triangulated sub-category $DMT_{\mathfrak{S}}(S(\mathfrak{S}))_F$ of $DM_{\mathbb{B},c}(S(\mathfrak{S}))_F$.

Recall [W4, Rem. 4.7] that thanks to *absolute purity* [CD1, Thm. 14.4.1], hypotheses (α) and (β) from Theorem 2.2 are satisfied as soon as the closures $\overline{S_\sigma}$ of all strata S_σ , $\sigma \in \mathfrak{S}$ are nilregular.

Definition 2.3. Let $S(\mathfrak{S}) = \coprod_{\sigma \in \mathfrak{S}} S_\sigma$ be a good stratification of a scheme $S(\mathfrak{S})$. Assume the following for all $\sigma \in \mathfrak{S}$: (α) S_σ is nilregular, (β) for any $i_\tau : S_\tau \hookrightarrow \overline{S_\sigma}$, the functor $i_\tau^!$ maps $\mathbf{1}_{\overline{S_\sigma}}$ to a Tate motive over S_τ . The category $DMT_{\mathfrak{S}}(S(\mathfrak{S}))_F$ of Theorem 2.2 is called the category of \mathfrak{S} -*constructible Tate motives over $S(\mathfrak{S})$* .

According to [W4, Thm. 4.5 (d)], the category $DMT_{\mathfrak{S}}(S(\mathfrak{S}))_F$ is pseudo-Abelian. Now let $S(\mathfrak{S}) = \coprod_{\sigma \in \mathfrak{S}} S_\sigma$ and $Y(\Phi) = \coprod_{\varphi \in \Phi} Y_\varphi$ be good stratifications of schemes $S(\mathfrak{S})$ and $Y(\Phi)$, respectively.

Definition 2.4. A morphism $\pi : S(\mathfrak{S}) \rightarrow Y(\Phi)$ is said to be a *morphism of good stratifications* if the pre-image $\pi^{-1}(Y_\varphi)$ of any stratum Y_φ of $Y(\Phi)$, $\varphi \in \Phi$ is a union of strata S_σ .

Definition 2.5. (a) A morphism $\pi : S(\mathfrak{S}) \rightarrow Y(\Phi)$ of good stratifications is said to be *of Abelian type* if it is proper, and if the following conditions are satisfied.

- (1) All strata Y_φ and S_σ , $\varphi \in \Phi$, $\sigma \in \mathfrak{S}$, are nilregular, and for any $i_\tau : S_\tau \hookrightarrow \overline{S_\sigma}$, the functor $i_\tau^!$ maps $\mathbf{1}_{\overline{S_\sigma}}$ to a Tate motive over S_τ .
- (2) For all $\sigma \in \mathfrak{S}$ such that S_σ is a stratum of $\pi^{-1}(Y_\varphi)$, the morphism $\pi_\sigma : S_\sigma \rightarrow Y_\varphi$ can be factorized,

$$\pi_\sigma = \pi'_\sigma \circ \pi''_\sigma : S_\sigma \xrightarrow{\pi''_\sigma} B_\sigma \xrightarrow{\pi'_\sigma} Y_\varphi ,$$

such that the motive

$$\pi''_{\sigma,*} \mathbf{1}_{S_\sigma} \in DM_{B,c}(B_\sigma)_F$$

belongs to the category $DMT(B_\sigma)_F$ of Tate motives over B_σ , the morphism π'_σ is proper and smooth, and its pull-back to any geometric point of Y_φ lying over a generic point is isomorphic to a finite disjoint union of Abelian varieties.

(b) Let Y be a scheme, equipped with a good stratification Φ with nilregular strata. Define the category $DM_{B,c,\Phi}^{Ab}(Y)_F$ as the strict, full, dense, F -linear triangulated sub-category of $DM_{B,c}(Y)_F$ generated by the images under π_* of the objects of $DMT_{\mathfrak{S}}(S(\mathfrak{S}))_F$, where $\pi : S(\mathfrak{S}) \rightarrow Y(\Phi)$ runs through the morphisms of Abelian type with target equal to $Y = Y(\Phi)$. Objects of $DM_{B,c,\Phi}^{Ab}(Y)_F$ are called *Φ -constructible motives of Abelian type over Y* .

(c) Let Y be a nilregular scheme. Set

$$DM_{B,c}^{Ab}(Y)_F := DM_{B,c,\Phi}^{Ab}(Y)_F ,$$

for the trivial stratification $\Phi = \{\varphi\}$.

Since the category $DM_{B,c}(Y)_F$ is pseudo-Abelian (see [H, Sect. 2.10]), so is $DM_{B,c,\Phi}^{Ab}(Y)_F$.

If a nilregular scheme Y is equipped with a good stratification Φ with nilregular strata, then by [W4, Thm. 4.5 (b)] and [CD1, Thm. 15.2.4], the category $DM_{B,c,\Phi}^{Ab}(Y)_F$ is closed under duality.

Next, we need to discuss weight structures.

Theorem 2.6. *Let Y be a scheme, equipped with a good stratification Φ with nilregular strata.*

(a) *The motivic weight structure w on $DM_{B,c}(Y(\Phi))_F$ ([H, Thm. 3.3], [B3,*

Thm. 2.1.1]) induces a weight structure, still denoted by the same letter w , on $DM_{\mathbb{B},c,\Phi}^{Ab}(Y)_F$.

(b) The motivic weight structure on $DM_{\mathbb{B},c,\Phi}^{Ab}(Y)_F$ is bounded.

Proof. We imitate the proof of [W4, Cor. 4.11]. Let \mathcal{K} be the strict, full, F -linear sub-category of $DM_{\mathbb{B},c}(Y(\Phi))_F$ of finite direct sums of motives isomorphic to images under π_* of objects in

$$CHMT_{\mathfrak{S}}(S(\mathfrak{S}))_F := DMT_{\mathfrak{S}}(S(\mathfrak{S}))_F \cap CHM(S(\mathfrak{S}))_F,$$

for morphisms $\pi : S(\mathfrak{S}) \rightarrow Y(\Phi)$ of Abelian type. Denote by \mathcal{D} the triangulated category generated by \mathcal{K} .

According to our definition and [W4, Cor. 4.12 (b)], \mathcal{D} equals the strict, full, F -linear triangulated sub-category of $DM_{\mathbb{B},c}(Y)_F$ generated by the images under π_* of the objects of $DMT_{\mathfrak{S}}(S(\mathfrak{S}))_F$, where $\pi : S(\mathfrak{S}) \rightarrow Y(\Phi)$ runs through the morphisms of Abelian type.

Following [W4, Rem. 4.4], the motivic weight structure induces a bounded weight structure on \mathcal{D} , whose heart contains \mathcal{K} .

Repeat the same argument with the pseudo-Abelian completion \mathcal{K}^{\natural} instead of \mathcal{K} . We get a bounded weight structure, induced by the motivic weight structure, on a triangulated sub-category \mathcal{D}^{\natural} of $DM_{\mathbb{B},c}(Y(\Phi))_F$.

Our claim is implied by [B1, Prop. 5.2.2], which states that \mathcal{D}^{\natural} is the pseudo-Abelian completion of \mathcal{D} , hence equal to $DM_{\mathbb{B},c,\Phi}^{Ab}(Y)_F$. **q.e.d.**

Definition 2.7. (a) Let Y be a scheme, equipped with a good stratification Φ with nilregular strata. A Φ -constructible Chow motive of Abelian type over Y is an object of

$$CHM_{\Phi}^{Ab}(Y)_F := DM_{\mathbb{B},c,\Phi}^{Ab}(Y)_{F,w=0}.$$

(b) Let Y be a nilregular scheme. Set

$$CHM^{Ab}(Y)_F := CHM_{\Phi}^{Ab}(Y)_F,$$

for the trivial stratification $\Phi = \{\varphi\}$.

Note that since $DM_{\mathbb{B},c,\Phi}^{Ab}(Y)_F$ is pseudo-Abelian, so is $CHM_{\Phi}^{Ab}(Y)_F$. Using [B1, Thm. 4.3.2 II], let us extract the following from the proof of Theorem 2.6.

Lemma 2.8. *Let Y be a scheme, equipped with a good stratification Φ with nilregular strata. Then the strict, full, pseudo-Abelian, F -linear sub-category $CHM_{\Phi}^{Ab}(Y)_F$ of $DM_{\mathbb{B},c}(Y)_F$ is generated by the images under π_* of objects in $CHMT_{\mathfrak{S}}(S(\mathfrak{S}))_F$, where $\pi : S(\mathfrak{S}) \rightarrow Y(\Phi)$ runs through the morphisms of Abelian type with target $Y(\Phi)$.*

In order to have the main result from [W4] at our disposal, let us check its hypotheses.

Proposition 2.9. *Let $\pi : S(\mathfrak{S}) \rightarrow Y(\Phi)$ be a morphism of Abelian type.
(a) Let $\sigma \in \mathfrak{S}$ such that S_σ is a stratum of $\pi^{-1}(Y_\varphi)$, and*

$$\pi_\sigma = \pi'_\sigma \circ \pi''_\sigma : S_\sigma \xrightarrow{\pi''_\sigma} B_\sigma \xrightarrow{\pi'_\sigma} Y_\varphi$$

*a factorization of the morphism $\pi_\sigma : S_\sigma \rightarrow Y_\varphi$ as in Definition 2.5 (a) (2).
Then the smooth Chow motive over Y_φ [L1, Def. 5.16]*

$$\pi'_{\sigma,*} \mathbf{1}_{B_\sigma} \in CHM^s(Y_\varphi)_F$$

is finite dimensional [AK, Déf. 9.1.1] (cmp. [Ki, Def. 3.7]).

(b) The morphism π satisfies the assumptions of [W4, Main Thm. 5.4].

Proof. The morphism π'_σ is proper and smooth, and its pull-back to any geometric point of Y_φ lying over a generic point is isomorphic to a finite disjoint union of Abelian varieties. According to [O'S, pp. 54–55], finite dimensionality can be checked after base change to the geometric generic points of Y_φ . Now apply [Kü, Thm. (3.3.1)]. This establishes part (a) of the claim.

But given Definition 2.5, part (a) is all that is needed in order to show that the assumptions of [W4, Main Thm. 5.4] are satisfied. **q.e.d.**

We thus get the following structural result.

Theorem 2.10. *Let Y be a scheme, equipped with a good stratification $Y = Y(\Phi)$ with nilregular strata. Then the F -category $CHM_\Phi^{Ab}(Y)_F$ of Φ -constructible Chow motive of Abelian type over Y is semi-primary (and pseudo-Abelian).*

Proof. Given [AK, Prop. 2.3.4 c)], our claim follows from Lemma 2.8, Proposition 2.9 (b) and [W4, Main Thm. 5.4]. **q.e.d.**

Here is our first result on conservativity in the motivic context.

Theorem 2.11. *Fix a generic point $\mathbf{Spec} k \hookrightarrow \mathbb{B}$ of the base scheme \mathbb{B} . Let $Y = Y(\Phi) = \coprod_{\varphi \in \Phi} Y_\varphi$ be a good stratification with nilregular strata, such that the generic points of all Y_φ lie over $\mathbf{Spec} k \hookrightarrow \mathbb{B}$. Denote by Y_k the base change of Y to $\mathbf{Spec} k$. Then the inverse image functor*

$$DM_{\mathbb{B},c,\Phi}^{Ab}(Y)_F \longrightarrow DM_{\mathbb{B},c,\Phi}^{Ab}(Y_k)_F$$

is conservative.

Proof. According to [CD1, Thm. 14.3.3], the categories $DM_{\mathbb{B},c}(\bullet)_F$ are *separated* in the sense of [CD1, Def. 2.1.7]. Therefore, it suffices to check the claim after application of the inverse image functors to all Y_φ . Given proper base change [CD1, Thm. 2.4.50 (4)], we are thus reduced to the case where the stratification Φ consists of a single stratum: $Y(\Phi) = Y_\varphi$. Recall that by assumption, the scheme Y_φ is nilregular.

According to Lemma 2.8 and [W4, Prop. 5.5], every object of $CHM_{\Phi}^{Ab}(Y)_F$ is a direct factor of a finite direct sum of objects isomorphic to $\pi'_* \mathbf{1}_B(p)[2p]$, for $p \in \mathbb{Z}$ and proper and smooth morphisms $\pi' : B \rightarrow Y_{\varphi}$.

In particular, $CHM_{\Phi}^{Ab}(Y)_F$ is contained in the category of smooth Chow motives over Y_{φ} . By assumption, the morphism $Y_{\varphi,k} \rightarrow Y_{\varphi}$ is dominant. Conservativity of the restriction of the inverse image

$$\alpha^* : DM_{B,c,\Phi}^{Ab}(Y)_F \longrightarrow DM_{B,c,\Phi}^{Ab}(Y_k)_F$$

to $CHM_{\Phi}^{Ab}(Y)_F$ thus follows from [O'S, pp. 54–55].

To treat the full triangulated category $DM_{B,c,\Phi}^{Ab}(Y)_F$, note that according to Theorem 2.6 (b), its weight structure is bounded.

By Theorem 2.10, the heart of the weight structure is semi-primary and pseudo-Abelian.

Furthermore, the functor α^* is weight exact.

According to [O'S, Prop. 5.1.1], the restriction of the functor α^* to the heart $CHM_{\Phi}^{Ab}(Y)_F$ is full.

Thus, the assumptions of Theorem 1.5 are all satisfied.

q.e.d.

3 Realizations

This section will be devoted to realizations. We fix a generic point $\mathbf{Spec} k$ of our base scheme \mathbb{B} , and assume that we are in one of the following situations.

- (i) k is embedded into \mathbb{C} via a morphism $\eta : k \hookrightarrow \mathbb{C}$, yielding a geometric point of \mathbb{B} , denoted by the same symbol

$$\eta : \mathbf{Spec} \mathbb{C} \longrightarrow \mathbf{Spec} k \hookrightarrow \mathbb{B}.$$

The *Betti realization* is defined in [Ay2, Déf. 2.1]. It is a family of covariant exact functors

$$R_{\eta,Z} : \mathbf{SH}(Z) \longrightarrow D(Z),$$

indexed by quasi-projective k -schemes Z . The source of $R_{\eta,Z}$ is the *stable homotopy category of Z -schemes* [Ay1, Sect. 4.5]. Its target is the derived category of the Abelian category of sheaves with values in Abelian groups on the topological space $Z(\mathbb{C})$ of points of Z with values in \mathbb{C} with respect to η . The functors $R_{\eta,Z}$ are symmetric monoidal [Ay2, Lemme 2.2]. According to [Ay2, Prop. 2.4, Thm. 3.4, Thm. 3.7], they commute with the functors f^* , f_* , $f_!$, $f^!$, provided the latter are applied to constructible objects (note that commutation holds without this restriction for the two functors f^* and $f_!$). In particular, they commute with Tate twists. In [CD1, Ex. 17.1.7], it is shown how to obtain from

the $R_{\eta,Z}$ a family of exact functors with analogous properties, and which we denote by the same symbols

$$R_{\eta,Z} : DM_{\mathbb{B},c}(Z) \longrightarrow D_c^b(Z) ,$$

where the right hand denotes the full triangulated sub-category of $D(Z)$ of classes of bounded complexes with constructible cohomology objects. The construction can be imitated to obtain F -linear versions of the Betti realization. Composing with the base change *via* $\mathbf{Spec} k \hookrightarrow \mathbb{B}$, we finally obtain a family of exact tensor functors

$$R_{\eta,X} : DM_{\mathbb{B},c}(X)_F \longrightarrow D_c^b(X_k)_F ,$$

still referred to as the Betti realization, and indexed by schemes X , whose base change $X_k := X \times_{\mathbb{B}} \mathbf{Spec} k$ is quasi-projective over k . The $R_{\eta,X}$ are symmetric monoidal; in particular, they respect the unit objects. They commute with the functors f^* , f_* , $f_!$, $f^!$ since $\mathbf{Spec} k \hookrightarrow \mathbb{B}$ is a projective limit of open immersions (use [CD1, Prop. 14.3.1]),

- (ii) k is of characteristic zero, and ℓ is a prime. The ℓ -adic realization is defined in [CD2, Sect. 7.2, see in part. Rem. 7.2.25]. It is a family of covariant exact functors

$$R_{\ell,Z} : DM_{\mathbb{B},c}(Z) \longrightarrow D_c^b(Z) ,$$

indexed by k -schemes Z of finite dimension. Its target is the bounded “derived category” of constructible \mathbb{Q}_{ℓ} -sheaves on Z [E, Sect. 6]. The functors $R_{\ell,Z}$ are symmetric monoidal, and they commute with the functors f^* , f_* , $f_!$, $f^!$ [CD2, Thm. 7.2.24]. In particular, they commute with Tate twists. The construction can be imitated to obtain F -linear versions of the ℓ -adic realization. Composing with the base change *via* $\mathbf{Spec} k \hookrightarrow \mathbb{B}$, we finally obtain a family of exact tensor functors

$$R_{\ell,X} : DM_{\mathbb{B},c}(X)_F \longrightarrow D_c^b(X_k)_F ,$$

still referred to as the ℓ -adic realization. The $R_{\ell,X}$ are symmetric monoidal, and they commute with the functors f^* , f_* , $f_!$, $f^!$.

In both settings, the categories $D_c^b(X_k)_F$ are equipped with a perverse t -structure; write $D^{t=0}(X_k)$ for its heart, $H^n : D_c^b(X_k)_F \rightarrow D^{t=0}(X_k)$, $n \in \mathbb{Z}$, for the cohomology functors, and

$$H^*R_X := (H^n R_X)_{n \in \mathbb{Z}} : DM_{\mathbb{B},c}(X)_F \longrightarrow \mathrm{Gr}_{\mathbb{Z}} D^{t=0}(X_k)$$

for the collection of all cohomology functors, preceded by the realization R_X . Here, we denote by $\mathrm{Gr}_{\mathbb{Z}} D^{t=0}(X_k)$ the \mathbb{Z} -graded category associated to the heart $D^{t=0}(X_k)$. We shall often refer to H^*R_X as the *cohomological realization*.

Remark 3.1. If k is finitely generated over \mathbb{Q} , then the ℓ -adic realization is a realization “with weights” as the action of local Frobenii allows for a

notion of purity. By contrast, there are no intrinsic weights on the target category of the Betti realization. The ideal solution would be to replace it by the *Hodge theoretical realization*

$$R_{\mathbf{H},\eta,X} : DM_{\mathbb{B},c}(X)_F \longrightarrow D^b(\mathbf{MHM}_{\mathbb{Q}}(X \times_{\eta} \mathbf{Spec} \mathbb{C}) \otimes_{\mathbb{Q}} F)$$

to the bounded derived category of *algebraic mixed Hodge modules* on $X \times_{\eta} \mathbf{Spec} \mathbb{C}$ [S, Sect. 4.2].

Let

$$R_{\bullet} : DM_{\mathbb{B},c}(\bullet)_F \longrightarrow D_c^b(\bullet_k)_F$$

be one of the two realizations considered above (Betti or ℓ -adic).

Proposition 3.2. *Let $Y = Y(\Phi) = \coprod_{\varphi \in \Phi} Y_{\varphi}$ be a good stratification with nilregular strata, such that the generic points of all Y_{φ} lie over $\mathbf{Spec} k \hookrightarrow \mathbb{B}$. In the context of the Betti realization, assume that Y_k is quasi-projective over $\mathbf{Spec} k$. Then the restriction of the cohomological realization functor on Y to $DM_{\mathbb{B},c,\Phi}^{Ab}(Y)_F$,*

$$H^*R_Y : DM_{\mathbb{B},c,\Phi}^{Ab}(Y)_F \longrightarrow \mathrm{Gr}_{\mathbb{Z}} D^{t=0}(Y_k)$$

satisfies assumptions (1), (2), (4) and (5) of Theorem 1.10. If R_{\bullet} is the ℓ -adic realization, then

$$H^*R_{\ell,Y} : DM_{\mathbb{B},c,\Phi}^{Ab}(Y)_F \longrightarrow \mathrm{Gr}_{\mathbb{Z}} D^{t=0}(Y_k)$$

also satisfies assumption (3) of Theorem 1.10.

Proof. Boundedness of the weight structure on $DM_{\mathbb{B},c,\Phi}^{Ab}(Y)_F$ is Theorem 2.6 (b). By Theorem 2.10, its heart $CHM_{\Phi}^{Ab}(Y)_F$ is semi-primary and pseudo-Abelian. According to [W4, Cor. 7.13] (see Lemma 2.8), the restriction of H^*R_Y to $DM_{\mathbb{B},c,\Phi}^{Ab}(Y)_F$ maps the radical to zero.

Thus, assumptions (1), (2) and (4) of Theorem 1.10 are met. Let us check assumption (5), *i.e.*, let us show that the zero motive is the only Chow motive in $CHM_{\Phi}^{Ab}(Y)_F$ whose realization is zero.

First, given Theorem 2.11, we may assume that \mathbb{B} equals the generic point $\mathbf{Spec} k$. Thus, we have $Y_k = Y$.

Second, by Definition 2.5 (b), and by [W4, Cor. 4.10 (b)], the triangulated category $DM_{\mathbb{B},c,\Phi}^{Ab}(Y)_F$ is obtained by successive gluing over the strata Y_{φ} of triangulated sub-categories $DM_{\mathbb{B},c}^{Ab}(Y_{\varphi})'_F$ of $DM_{\mathbb{B},c}^{Ab}(Y_{\varphi})_F$. The $DM_{\mathbb{B},c}^{Ab}(Y_{\varphi})'_F$ inherit the weight structure (Theorem 2.6) from $DM_{\mathbb{B},c}^{Ab}(Y_{\varphi})_F$, and according to Theorem 2.10, their hearts are semi-primary and pseudo-Abelian. Thus, the abstract theory of *intermediate extensions* can be applied: according to [W4, Summ. 2.12], any object of $CHM_{\Phi}^{Ab}(Y)_F$ is a direct sum of Chow motives of the form $j_{\varphi,!} M_{\varphi}$, for certain Chow motives $M \in CHM_{\Phi}^{Ab}(Y_{\varphi})_F$. Here, $j_{\varphi,!}$ denotes the intermediate extension [W4, Def. 2.10] associated to the immersion $j_{\varphi} : Y_{\varphi} \hookrightarrow Y$. If the cohomological realization $H^*R_{Y_{\varphi}}(M) \in \mathrm{Gr}_{\mathbb{Z}} D^{t=0}(Y_{\varphi})$ is zero, then so are the cohomological realizations of all $j_{\varphi,!} M_{\varphi}$,

hence of all M_φ since R_\bullet is compatible with inverse images. Thus, we may assume that the stratification Φ consists of a single (nilregular) stratum: $Y(\Phi) = Y_\varphi$.

According to Lemma 2.8, [W4, Prop. 5.5] and Proposition 2.9, any Chow motive M in $CHM^{Ab}(Y_\varphi)_F$ is smooth and finite dimensional. The same is therefore true for its pull-back M_ξ to any generic point ξ of Y_φ .

So if we assume the cohomological realization $H^*R_{Y_\varphi}(M) \in \text{Gr}_{\mathbb{Z}} D^{t=0}(Y_\varphi)$ to be zero, then the realization of any M_ξ is zero, again since R_\bullet is compatible with inverse images. Therefore [Ki, Cor. 7.3], all M_ξ are zero. But according to [O'S, pp. 54–55], this implies that M is zero.

If R_Y is the ℓ -adic realization, then hypothesis (3) of Theorem 1.10 is also met, *i.e.*, the morphisms in the image of $H^*R_{\ell,Y}$ are strict with respect to the weight filtration [B4, Thm. 2.5.4 (II) (1), Prop. 1.3.2 (II) (2)]. Note that Y_k is of finite type over k , hence *very reasonable* in the sense of [B4, Def. 2.1.1 (4)]. **q.e.d.**

Here is our second result on conservativity in the motivic context.

Theorem 3.3. (a) *Let $Y = Y(\Phi) = \coprod_{\varphi \in \Phi} Y_\varphi$ be a good stratification with nilregular strata, such that the generic points of all Y_φ lie over $\mathbf{Spec} k \hookrightarrow \mathbb{B}$. In the context of the Betti realization, assume that Y_k is quasi-projective over $\mathbf{Spec} k$. Then the restriction of the realization functor on Y to $DM_{\mathbb{B},c,\Phi}^{Ab}(Y)_F$,*

$$R_Y : DM_{\mathbb{B},c,\Phi}^{Ab}(Y)_F \longrightarrow D_c^b(Y_k)_F ,$$

is conservative.

(b) *Assume that $\mathbb{B} = \mathbf{Spec} k$. Let $Y = Y(\Phi) = \coprod_{\varphi \in \Phi} Y_\varphi$ be a good stratification with nilregular strata. In the context of the Betti realization, assume that Y is quasi-projective over $\mathbf{Spec} k$. Then the restriction of the realization functor on Y to $DM_{\mathbb{B},c,\Phi}^{Ab}(Y)_F$,*

$$R_Y : DM_{\mathbb{B},c,\Phi}^{Ab}(Y)_F \longrightarrow D_c^b(Y)_F ,$$

is conservative.

Proof. (b) is a special case of (a) (put $\mathbb{B} = \mathbf{Spec} k$).

For the ℓ -adic realization, Theorem 1.10 can be applied directly, thanks to Proposition 3.2.

Let us treat the case when R_\bullet is the Betti realization. Given our present state of knowledge, we may suppose, but do not know the analogue of hypothesis (3) of Theorem 1.10 to hold; therefore, we need an alternative approach.

As earlier (Theorem 2.11 and its proof), we may assume that \mathbb{B} equals the generic point $\mathbf{Spec} k$, and that the stratification of $Y(\Phi)$ consists of a single (nilregular) stratum: $Y(\Phi) = Y_\varphi$.

Now recall from the proof of Theorem 1.10 that hypothesis (3) is only used *via* Lemma 1.9. The idea therefore consists in deducing Lemma 1.9 from the little we know. By Lemma 2.8, [W4, Prop. 5.5] and Proposition 2.9, the

category $CHM^{Ab}(Y_\varphi)_F$ consists of Chow motives which are smooth over Y_φ . As it generates the triangulated category $DM_{B,c}^{Ab}(Y_\varphi)_F$ (Theorem 2.6 (b)), the cohomological realization of any object of the latter gives perverse sheaves which are actually local systems (up to a shift). This holds in particular for the objects X and Z occurring in Lemma 1.9; therefore, the effect of the cohomological realization of a morphism between them can be read off the restriction of the latter to the generic points of Y_φ , where comparison with the ℓ -adic realization is available. **q.e.d.**

The third and main result on conservativity reads as follows; it generalizes the ℓ -adic version of [W3, Thm. 1.13].

Theorem 3.4. *Assume k to be of characteristic zero, and let ℓ a prime. Let $Y = Y(\Phi) = \coprod_{\varphi \in \Phi} Y_\varphi$ be a good stratification with nilregular strata, such that the generic points of all Y_φ lie over $\text{Spec } k \hookrightarrow \mathbb{B}$. Then the ℓ -adic realization $R_{\ell,Y}$ respects and detects the weight structure on $DM_{B,c,\Phi}^{Ab}(Y)_F$. More precisely, let $M \in DM_{B,c,\Phi}^{Ab}(Y)_F$, and $\alpha \leq \beta$ two integers.*

(a) *M lies in the heart $CHM_\Phi^{Ab}(Y)_F$ of w if and only if the n -th perverse cohomology object $H^n R_{\ell,Y}(M) \in D^{t=0}(Y_k)$ of $R_{\ell,Y}(M)$ is pure of weight n , for all $n \in \mathbb{Z}$.*

(b) *M lies in $DM_{B,c,\Phi}^{Ab}(Y)_{F,w \leq \alpha}$ if and only if $H^n R_{\ell,Y}(M)$ is of weights $\leq n + \alpha$, for all $n \in \mathbb{Z}$.*

(c) *M lies in $DM_{B,c,\Phi}^{Ab}(Y)_{F,w \geq \beta}$ if and only if $H^n R_{\ell,Y}(M)$ is of weights $\geq n + \beta$, for all $n \in \mathbb{Z}$.*

(d) *M is without weights $\alpha, \alpha + 1, \dots, \beta$ if and only if $H^n R_{\ell,Y}(M)$ is without weights $n + \alpha, n + \alpha + 1, \dots, n + \beta$, for all $n \in \mathbb{Z}$.*

Proof. According to Proposition 3.2, the assumptions of Theorem 1.11 are satisfied for $\mathcal{H} = H^0 R_{\ell,Y}$. **q.e.d.**

Remark 3.5. If k is finitely generated over \mathbb{Q} , then there is an intrinsic notion of weights on those objects of the heart $D^{t=0}(Y_k)$ of the perverse t -structure on $D_c^b(Y_k)_F$, which are in the image of the cohomological realization [B4, Prop. 2.5.1 (II)].

In general, the weights of $H^* R_{\ell,Y}(M)$ are by definition those induced by the weight filtration of the functor $H^* R_{\ell,Y}$ as considered in the previous section (these coincide with the above when k is finitely generated over \mathbb{Q}).

Remark 3.6. The analogue of Theorem 3.4 should hold for the Betti, and the Hodge theoretical realization. In the absence of the latter, and/or a general comparison statement between the Betti and the ℓ -adic realization, the problem for the Betti realization consists in the verification of assumption (3) of Theorem 1.10 (which *directly* enters the proof of Theorem 1.11). Contrary to the proof of Theorem 3.3, reduction to a statement on individual strata of $Y(\Phi)$ does not seem to work.

Let us spell out the special case $\mathbb{B} = \mathbf{Spec} k$ of Theorem 3.4.

Corollary 3.7. *Let k be a field of characteristic zero, ℓ a prime, Y a scheme over $\mathbb{B} = \mathbf{Spec} k$, and $Y = Y(\Phi) = \coprod_{\varphi \in \Phi} Y_{\varphi}$ a good stratification with nilregular strata. Let $M \in DM_{\mathbb{B}, c, \Phi}^{Ab}(Y)_F$, and $\alpha \leq \beta$ two integers.*

(a) *M lies in the heart $CHM_{\Phi}^{Ab}(Y)_F$ of w if and only if the n -th perverse cohomology object $H^n R_{\ell, Y}(M) \in D^{t=0}(Y)$ of $R_{\ell, Y}(M)$ is pure of weight n , for all $n \in \mathbb{Z}$.*

(b) *M lies in $DM_{\mathbb{B}, c, \Phi}^{Ab}(Y)_{F, w \leq \alpha}$ if and only if $H^n R_{\ell, Y}(M)$ is of weights $\leq n + \alpha$, for all $n \in \mathbb{Z}$.*

(c) *M lies in $DM_{\mathbb{B}, c, \Phi}^{Ab}(Y)_{F, w \geq \beta}$ if and only if $H^n R_{\ell, Y}(M)$ is of weights $\geq n + \beta$, for all $n \in \mathbb{Z}$.*

(d) *M is without weights $\alpha, \alpha + 1, \dots, \beta$ if and only if $H^n R_{\ell, Y}(M)$ is without weights $n + \alpha, n + \alpha + 1, \dots, n + \beta$, for all $n \in \mathbb{Z}$.*

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